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SPLITTING SQUARES*

BY

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ABSTRACT

We show that for every Borel function $f\colon [2^\omega]^n \to 2^\omega$ there exists a closed countably splitting family A such that $f \restriction [A]^n$ omits a perfect set of values in 2^ω.

1. Introduction and notation

During the Luminy Set Theory Workshop 2004 Jindrich Zapletal asked the following question:

> Does there exist a Borel function $f\colon [2^\omega]^2 \to 2^\omega$ such that for every analytic splitting family $A \subseteq 2^\omega$ the restriction $f \restriction [A]^2$ maps onto 2^ω.

Using the arrow notation from partition calculus one might formalize this question as follows: Does the relation

$$2^\omega \not\to_{Borel} [analytic\ splitting]^2_{2^{\aleph_0}}$$

hold? Part of the motivation for asking this question was certainly Velickovic's result in $[VW]$ that

$$\omega^\omega \not\to_{Borel} [analytic\ unbounded]^3_{2^{\aleph_0}}$$

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holds. On the other hand, it follows from the canonization of all Borel functions $f\colon (\omega^\omega)^2 \to 2^\omega$ given in [Sp1] that

$$\omega^\omega \to_{Borel} [\textit{closed unbounded}]^2_{2^{\aleph_0}}$$

holds, i.e., for every Borel function $f\colon [\omega^\omega]^2 \to 2^\omega$ there exists a closed unbounded family A such that $f \restriction [A]^2$ is not onto, and actually $f \restriction [A]^2$ omits a perfect set of colors. In this paper we prove the positive partition relation

$$2^\omega \to_{Borel} [\textit{closed countably splitting}]^n_{2^{\aleph_0}}$$

for every n, and thus answer Zapletal's question negatively.

Recall that $A \subseteq 2^\omega$ is a **splitting family** iff for every $a \in [\omega]^\omega$ there exists $x \in A$ that **splits** a, i.e. $\exists^\infty n \in a \exists^\infty m \in a(x(n) = 0 \wedge x(m) = 1)$. If, moreover, for every countable $F \subseteq [\omega]^\omega$ we can find $x \in A$ that splits every member of F, A is called **countably splitting**. As usual, 2^ω is equipped with the Cantor space topology, and $[2^\omega]^n$ (the set of n-element subsets of 2^ω) is identified with the set of increasing elements of $(2^\omega)^n$, i.e.

$$\{\overline{x} \in (2^\omega)^n : \forall i < n-1 x_i <_{lex} x_{i+1}\}$$

(where $<_{lex}$ is the lexicographic order), and hence it is a subspace of $(2^\omega)^n$. Analytic countably splitting families have been studied in [Sp2]. In that paper I showed that every analytic countably splitting family contains a closed countably splitting family. For this I isolated the notion of a splitting tree. Later I noticed that splitting tree forcing is a dense suborder of a proper forcing $\mathcal{S}$ that has been introduced by Shelah in [Sh].

Let us define $\mathcal{S}$. It is the set of all trees $p \subseteq 2^{<\omega}$ such that there exists a function $K^p\colon p \to \omega$ with the property that for every $\nu \in p$, for every $i \geqslant K(\nu)$ and $j < 2$ there exists an extension μ of ν in p such that $|\mu| > i$ and $\mu(i) = j$. The order on $\mathcal{S}$ is inclusion. It is easy to see that $[p]$ (the set of branches of p) is countably splitting for every $p \in \mathcal{S}$.

If $p \subseteq 2^{<\omega}$ is any tree and $i < \omega$, we let $p \restriction i = \{\nu \in p : |\nu| = i\}$ and $p[i] = \{\nu(i)\colon \nu \in p \restriction i+1\}$. By stem$(p)$ or st(p) we denote the **stem** of p, i.e. its shortest splitnode. If $\nu \in p$ then $p(\nu) = \{\mu \in p : \mu \subseteq \nu \vee \nu \subseteq \mu\}$. Our main result will follow with some extra work from the canonization of Borel functions $f\colon (2^\omega)^n \to 2^\omega$ modulo restriction to blocks $\prod_{i<n}[p_i]$ where $\overline{p} = \langle p_0, \ldots, p_{n-1}\rangle \in \mathcal{S}^n$. More precisely, we shall prove the following (see Theorem 10 (b) below):

THEOREM: *Given any Borel function* $f\colon (2^\omega)^n \to 2^\omega$ *and* $\overline{p} \in \mathcal{S}^n$, *there exist* $\overline{q} \leqslant \overline{p}$ *(i.e.,* $q_i \leqslant p_i$ *for all* $i < n$*) and* $E \subseteq n$ *such that for all* $\overline{x}, \overline{y} \in \prod_{i<n}[q_i]$ *we have*

$$f\overline{x} = f\overline{y} \Leftrightarrow \{x_i : i \in E\} = \{y_i : i \in E\}.$$

Let us briefly comment on the history of canonization results and what they are about. For a given class of functions on a certain space one would like to find a list of canonical (i.e. naturally definable, easily understandable) functions, such that every given function induces the same equivalence relation as some canonical function on some reasonably large and definable subdomain. The first example of such a result is a theorem of Erdős and Rado [ER] which canonizes functions on $[\omega]^k$ for some finite k. Ramsey's Theorem [R], which deals only with such functions with finite range is a special case of it. Very similarly as in our Theorem above, the canonical functions in the Erdös–Rado Theorem are the generalized projections. Stepping to uncountable spaces, one quickly realizes that some regularity requirement (i.e. measurability in some sense) has to be imposed on the functions in order to make the canonization project look reasonable. And indeed, there is a long history of such results, and it is a typical phenomenon, that a new canonization will generalize many others. As an example we refer to our result in [KlSp], where also a more detailed account of canonization results is given.

Our Theorem above is a strengthening of a corollary of a result of Lefmann. In [L], he canonized Borel functions $f\colon [2^\omega]^n \to 2^\omega$ modulo restrictions to perfect cubes, i.e. sets of the form $[p]^n$ with $p \subseteq 2^{<\omega}$ a perfect tree. As a corollary one easily obtains that the generalized projections (as in our Theorem) canonize the class of Borel functions $f\colon (2^\omega)^n \to 2^\omega$ modulo restriction to perfect blocks (sets of the form $\prod_{i<n}[p_i]$, where each p_i is perfect). As splitting trees are perfect, our result implies this. A direct analog of Lefmann's result for splitting trees does not hold. Part of the reason is that in contrast to perfect trees there exists $p \in \mathcal{S}$ and a coloring of the splitnodes of p by two colors such that for no $q \leqslant p, q \in \mathcal{S}$, all the splitnodes of q have the same color (see Theorem 14). Then, clearly, there is a continuous $f\colon [[p]]^2 \to 2$ such that for no $q \leqslant p, q \in \mathcal{S}, f \restriction [[q]]^2$ is constant.

As I explained in the introduction of [Sp1] there exists a close connection between a canonization-modulo-blocks result as in the theorem above and the description of the lattice of constructibility degrees of reals in a forcing extension of $\mathbf{L}$ by a finite power of the relevant tree forcing. That is why the main body of this paper consists of showing that in $\mathbf{L}^{\mathcal{S}^n}$ the reals have the minimal possible degree structure (see Theorem 10(a) below):

THEOREM′: *The lattice of constructibility degrees of reals in* $\mathbf{L}^{\mathcal{S}^n}$ *is isomorphic to* $(\mathcal{P}(n), \subseteq)$.

Whereas the Theorem above is an actual strengthening of Lefmann's Corollary, parts of our proof of Theorem′ give the same result for a finite power of Sacks forcing, i.e., a finite power of Sacks forcing produces the minimal degree structure. This result has been known to several set theorists for a long time, but it seems to be impossible to find its origin and, even more, to give a reference where it appeared first.

I end this introduction by establishing some notation for the product forcing $\mathcal{S}^n$. Given $\overline{p} \in \mathcal{S}^n$ and $j < \omega$ we let $\overline{p} \restriction j = \langle p_i \restriction j : i < n \rangle$; occasionally we identify $\overline{p} \restriction j$ with $\prod_{i<n} p_i \restriction j$, especially when we write $\overline{\sigma} \in \overline{p} \restriction j$ etc. Given $\overline{\sigma} \in \prod_{i<n} p_i$, we let $\overline{p}(\overline{\sigma}) = \langle p_i(\sigma_i) : i < n \rangle$. Here and above we tacitly denoted the ith coordinate of some n-tupel $\overline{t}$ by t_i. Occasionally, when t comes already with a number of indices we will write $\overline{t}_i$ or $(\overline{t})_i$ instead of t_i. As a last piece of definition we need that of a **front** of a tree $p \in \mathcal{S}$. It is a subset F of p such that any two elements are incompatible, i.e. none is an extension of the other, and every branch of p has an initial segment that belongs to F. By compactness, such F must be finite. We say that a front F **refines** another front F', if every element of F extends an element of F'. Finally, given any sequence σ we let $F \restriction \sigma$ denote the set of elements of F that are compatible with σ.

2. The degree structure of $\mathcal{S}^n$

Our first Lemma shows that $\mathcal{S}^n$ has continuous reading of names. For $\mathcal{S}$ this is part of the proof of [Sh, Proposition 2.10].

LEMMA 1: *Let* $\overline{p} \in \mathcal{S}^n$*, let* $\dot{\alpha}$ *be an* $\mathcal{S}^n$*-name for an ordinal and* $r < \omega$*. There exist* $\overline{q} \leqslant \overline{p}$ *and* $k < \omega$ *such that*

(1) $\overline{q} \restriction r = \overline{p} \restriction r$;
(2) $\forall i < n \forall j < \omega \; q_i[j] = p_i[j]$;
(3) *for every* $\overline{\sigma} \in \overline{q} \restriction k$ *there exists* $\alpha_{\overline{\sigma}} \in \mathbf{Ord}$ *such that* $\overline{q}(\overline{\sigma}) \Vdash \dot{\alpha} = \alpha_{\overline{\sigma}}$.

Proof: Let $\langle \pi_l : l < 2^n \rangle$ enumerate ${}^n 2$. We define sequences $\langle \overline{p}^l : l \leqslant 2^n \rangle$ in $\mathcal{S}^n$ and $\langle n_l : l \leqslant 2^n \rangle$ in ω such that for all $l < 2^n$ we have:

(i) $\overline{p}^0 = \overline{p}, \overline{p}^{l+1} \leqslant \overline{p}^l$,
(ii) $\overline{p}^{l+1} \restriction n_l = \overline{p}^l \restriction n_l$ and $\forall i < n \forall j \geqslant n_l p_i^{l+1}[j] = p_i^l[j]$
(iii) $n_0 = r, n_l < n_{l+1}$ and for every $\overline{\sigma} \in \langle p_i^{l+1}(st(p_i)^\frown \pi_l(i)) : i < n \rangle \restriction n_l$ there exists $\alpha_{\overline{\sigma}}$ such that $\overline{p}^{l+1}(\overline{\sigma}) \Vdash \dot{\alpha} = \alpha_{\overline{\sigma}}$.

Suppose we have $\overline{p}^l$ and n_l for $l < 2^n$ as desired. Fix $\overline{\tau} \in \langle p_i^l(st(p_i)^\frown 1-\pi_l(i)) : i < n\rangle \restriction n_l$ arbitrarily.

Define $n_{l+1} = \max\{K^{p_i^l}(\tau_i) : i < n\}$ and let $p_i^{l+1}(st(p_i)^\frown 1 - \pi_l(i)) = p_i^l(st(p_i)^\frown 1 - \pi_l(i))$ and $p_i^{l+1}(st(p_i)^\frown \pi_l(i)) \restriction n_{l+1} = p_i^l(st(p_i)^\frown \pi_l(i)) \restriction n_{l+1}$ for every $i < n$. Note that this already guarantees (ii). It is easy to find $\overline{p}' \leqslant \langle p_i^l(st(p_i)^\frown \pi_l(i)) : i < n\rangle$ such that $\overline{p}' \restriction n_{l+1} = \langle p_i^l(st(p_i)^\frown \pi_l(i)) : i < n\rangle \restriction n_{l+1}$ and (iii) holds for $\overline{p}'$ in place of $\overline{p}^{l+1}$. Thus we define $p_i^{l+1}(st(p_i)^\frown \pi_l(i))$ as p_i' and so the construction is completed as desired. Now clearly we can let $\overline{q} = \overline{p}_{2^n}$ and $k = n_{2^n}$ and we are done. ∎

Completely the same argument as for Lemma 1 gives the following variant of it:

Lemma 1′: *Let $\overline{p} \in \mathcal{S}^n$, let $D \subseteq \mathcal{S}^n$ be dense and $r < \omega$. There exist $\overline{q} \leqslant \overline{p}$ and $k < \omega$ such that:*

(1) $\overline{q} \restriction r = \overline{p} \restriction r$;

(2) $\forall i < n \forall j < \omega\ q_i[j] = p_i[j]$;

(3) *for every $\overline{\sigma} \in q \restriction k$ we have $q(\overline{\sigma}) \in D$.*

Corollary 2: *Let $\overline{p} \in \mathcal{S}^n$, $\mathcal{S}^n$-name $\dot{x}$ and $r < \omega$ be given such that $\overline{p} \Vdash \dot{x} \in 2^\omega$. There exist $\overline{q} \leqslant \overline{p}$, an increasing sequence $\langle n_k : k < \omega\rangle$ in ω and a family $\langle \xi_{\overline{\sigma}} : \overline{\sigma} \in \bigcup_{k<\omega} \overline{q} \restriction n_k\rangle$ in $2^{<\omega}$ such that $\overline{q} \restriction r = \overline{p} \restriction r$ and for every k and $\overline{\sigma} \in \overline{q} \restriction n_k$ we have $\overline{q}(\overline{\sigma}) \Vdash \dot{x} \restriction k = \xi_{\overline{\sigma}}$.*

Proof: By Lemma 1 it is straightforward to construct a descending sequence $\langle \overline{q}^k : k < \omega\rangle$ below $\overline{p}$, an increasing sequence $\langle n_k : k < \omega\rangle$ in ω and a family $\langle \xi_{\overline{\sigma}} : \overline{\sigma} \in \bigcup \overline{q}^k \restriction n_k\rangle$ in $2^{<\omega}$ such that we have:

(1) $n_0 = r$, $\overline{q}_0 = \overline{p}$, $\overline{q}^{k+1} \restriction n_k = \overline{q}^k \restriction n_k$;

(2) $\forall \overline{\sigma} \in \overline{q}^k \restriction n_k \forall i < n \forall j \geqslant n_k\ q_i^{k+1}(\sigma_i)[j] = q_i^k(\sigma_i)[j]$;

(3) $\forall \overline{\sigma} \in \overline{q}^k \restriction n_k\ \overline{q}^k(\overline{\sigma}) \Vdash \dot{x} \restriction k = \xi_{\overline{\sigma}}$.

Let $\overline{q}$ be determined by $\overline{q} \restriction n_k = \overline{q}^k \restriction n_k$ for all $k < \omega$. For $i < n$ we define $K^{q_i}\colon q_i \to \omega$ as follows: Given $\nu \in q_i$, choose k minimal such that $|\nu| \leqslant n_k$ and $\sigma \in q_i \restriction n_k$ extending ν. Define $K^{q_i}(\nu) = K^{q_i^k}(\sigma)$. Then K^{q_i} witnesses that $q_i \in \mathcal{S}$ by (2). ∎

Remark: Given N a countable transitive model of ZF^- and $\overline{p} \in N \cap \mathcal{S}^n$ it is straightforward, using the proof of Corollary 2, to construct in V a descending sequence $\langle \overline{q}_k : k < \omega\rangle$ below $\overline{p}$ such that each $\overline{q}_k$ belongs to N, every $\mathcal{S}^n$-name in N for an ordinal gets decided by some $\overline{q}_k$ and, letting $\overline{q} = \bigcap_{k<\omega} \overline{q}_k$, we have that $\overline{q} \in \mathcal{S}^n$ and every $\overline{x} \in \prod_{i<n}[q_i]$ is $\mathcal{S}^n$-generic over N.

The next lemma will imply that the forcing $\mathcal{S}$ adds a minimal degree of constructibility. The idea of its proof will later be combined with arguments which work for finite products of many other compact tree forcings as well. That is why we retain this Lemma although it follows from the Main Lemma below.

LEMMA 3: *Suppose that $p_0 \in \mathcal{S}$ and $\dot{x}$ is an $\mathcal{S}$-name such that*

$$p_0 \Vdash \dot{x} \in 2^\omega \setminus V.$$

There exist $q \in \mathcal{S}$, a front F of q and a family $\langle \xi_\sigma : \sigma \in F \rangle$ such that

(1) *$q \leqslant p_0$ and $q[i] = p_0[i]$ holds for every $i < \omega$;*
(2) *ξ_σ and $\xi_{\sigma'}$ are incompatible elements of $2^{<\omega}$ for any different $\sigma, \sigma' \in F$;*
(3) *$q(\sigma) \Vdash \xi_\sigma \subseteq \dot{x}$ for all $\sigma \in F$.*

Proof: For $p \in \mathcal{S}$ and $\sigma \in p$ let $TP(p, \sigma)$ be the tree of possibilities for $\dot{x}$ below $p(\sigma)$, i.e. $TP(p, \sigma) = \{\xi \in 2^{<\omega} : \exists q \leqslant p(\sigma) q \Vdash \xi \subseteq \dot{x}\}$. By our assumption we easily conclude that $TP(p, \sigma)$ is a perfect tree for every $p \leqslant p_0$ and $\sigma \in p$.

Let K: $p_0 \to \omega$ witness that $p_0 \in \mathcal{S}$. Fix $\tau \in p_0(stem(p_0)^\frown 0)$ of length $K(stem(p_0)^\frown 0)$. By the remark above we can choose branches $x_\tau \in [TP(p_0, \tau)]$ and $x_\sigma \in [TP(p_0, \sigma)]$, for every $\sigma \in p_0(stem(p_0)^\frown 1) \restriction K(stem(p_0)^\frown 0)$ which are pairwise distinct. Let $S = p_0(stem(p_0)^\frown 1) \restriction K(stem(p_0)^\frown 0)$. By Corollary 2 we may assume that for every n there is k such that for every $\sigma \in p_0 \restriction k, p_0(\sigma)$ decides $\dot{x} \restriction n$. Thus we can first choose n large enough so that $x_\tau \restriction n$ and $x_\sigma \restriction n$ for $\sigma \in S$ are all different and then find $k \geqslant K(stem(p_0)^\frown 0)$ and extensions τ' of τ and σ' of σ in p_0, all of length k, so that $p_0(\tau') \Vdash \dot{x} \restriction n = x_\tau \restriction n$ and $p_0(\sigma') \Vdash \dot{x} \restriction n = x_\sigma \restriction n$, and moreover $x_\tau \in [TP(p_0, \tau')]$ and $x_\sigma \in [TP(p_0, \sigma')]$ for every $\sigma \in S$.

Next we fix $x_\rho \in [TP(p_0, \rho)]$ for every $\rho \in p_0(stem(p_0)^\frown 0)$ of length $K(\tau')$ which is incompatible with τ', such that they are pairwise different and also different from all the x_σ and x_τ chosen above. Let R be the set of all ρ just described.

Now we choose a new n large enough so that the mapping $\nu \mapsto x_\nu \restriction n$ is one-to-one on $S \cup R$. Then we find a new $k \geqslant K(\tau')$, for each $\sigma \in S$ an extension σ'' of σ' of length k in p_0 and for each $\rho \in R$ an extension ρ' of length k in p_0 such that $p_0(\sigma'') \Vdash \dot{x} \restriction n = x_\sigma \restriction n$ and $p_0(\rho') \Vdash \dot{x} \restriction n = x_\rho \restriction n$ and $x_\rho \in [TP(p_0, \rho')]$. Let K^* be minimal among the $K(\sigma'')$ where $\sigma \in S$. Finally let $T = p_0(\tau') \restriction K^*$. For each $\mu \in T$ choose $x_\mu \in [TP(p_0, \mu)]$ such that the mapping $\nu \mapsto x_\nu$ is one-to-one on $R \cup T$. Choose again a new n such that $\nu \mapsto x_\nu \restriction n$ is one-to-one on $R \cup T$ and then a new and final $k \geqslant K$ and extensions ρ'' of ρ' in p_0 for all

$\rho \in R$ and μ' of μ in p_0 for all $\mu \in T$, such that all ρ'' and μ' have length k and $p_0(\mu') \Vdash x_\mu \restriction n \subseteq \dot{x}$ and $p_0(\rho'') \Vdash x_\rho \restriction n \subseteq \dot{x}$ hold.

Now we let $F = \{\sigma'' : \sigma \in S\} \cup \{\rho'' : \rho \in R\} \cup \{\mu' : \mu \in T\}$ and $q = \bigcup\{p_0(\nu) : \nu \in F\}$. For each $\nu \in F$ we let $\xi_\nu \in 2^{<\omega}$ be maximal such that $p_0(\nu) \Vdash \xi_\nu \subseteq \dot{x}$. Properties (2) and (3) can now easily be checked. Moreover $q[i] = p_0[i]$ follows from our construction. Indeed, firstly we have $q \restriction K(stem(p_0)^\frown 0) = p_0 \restriction K(stem(p_0)^\frown 0)$. Secondly, $q(stem(p_0)^\frown 0) \restriction K(\tau') = p_0(stem(p_0)^\frown 0) \restriction K(\tau')$. Hence, by definition of K we have $q[i] = q(stem(p_0)^\frown 0)[i] = \{0,1\}$ and thus $q[i] = p[i]$ for every i with $K(stem(p_0)^\frown 0) \leqslant i < K(\tau')$. Thirdly, we have $q(\tau') \restriction K^* = p_0(\tau') \restriction K^*$. As $p_0(\tau')[i] = \{0,1\}$ for every $i \geqslant K(\tau')$ we conclude $q[i] = p_0[i]$ for every i with $K(\tau') \leqslant i < K^*$. Finally, if $\sigma \in S$ is such that $K^* = K(\sigma'')$, then $p_0(\sigma'')[i] = \{0,1\}$ for every $i \geqslant K(\sigma'')$. As $q(\sigma'') = p_0(\sigma'')$ we are done. ∎

Corollary 4: *Suppose that $p_0 \in \mathcal{S}$ and $\dot{x}$ is an $\mathcal{S}$-name such that $p_0 \Vdash \dot{x} \in 2^\omega \setminus V$. There exist $q \in \mathcal{S}, q \leq p_0$, a sequence $\langle F_k : k < \omega\rangle$ of fronts in q and a sequence $\langle \xi_\sigma : \sigma \in \bigcup_{k<\omega} F_k\rangle$ in $2^{<\omega}$ such that the following hold:*

1. *F_{k+1} strictly refines F_k;*
2. *ξ_σ is longest possible such that $q(\sigma) \Vdash \xi_\sigma \subseteq \dot{x}$, for every $\sigma \in \bigcup_{k<\omega} F_k$;*
3. *the family $\langle \xi_\tau : \tau \in q(\sigma) \cap F_{k+1}\rangle$ is pairwise incompatible, for every $\sigma \in F_k$.*

(Note that (3) implies that $\langle \xi_\sigma : \sigma \in F_k\rangle$ is pairwise incomptible for every k.)

Proof: We construct sequences $\langle q_n : n < \omega\rangle$ in $\mathcal{S}$, $\langle \xi_\sigma : \sigma \in F_k\rangle$ in $2^{<\omega}$ and $\langle F_k : k < \omega\rangle$ such that F_k is a front of q_k, recursively, as follows:

Let $q_0 = p_0, F_0 = \{stem(p_0)\}$ and $\xi_\emptyset$ be such that (2) holds, possibly $\xi_\emptyset = \emptyset$. Suppose we have gotten up to k. For each $\tau \in F_k$ we apply Lemma 3 with $q_k(\tau)$ in place of p_0. Thus we obtain F^τ, q^τ and a family $X^\tau = \langle \xi_\sigma : \sigma \in F^\tau\rangle$ such that the respective (1), (2), (3) hold. Now let $q_{k+1} = \bigcup\{q^\tau : \tau \in F_k\}$ and $F_{k+1} = \bigcup\{F^\tau : \tau \in F_k\}$. This finishes our recursion. By construction we have that $\bigcap_{k<\omega} q_k$ is the tree generated by $\bigcup_{k<\omega} F_k$. This is the desired q. For this we only have to check that $q \in \mathcal{S}$. As in the inductive step of the construction we always chose q^τ with property (1) from Lemma 3 (with $q_k(\tau)$ in place of p_0), we conclude that for each k and $\tau \in F_k$, for every $m > k$ we have $q_m(\tau)[i] = q_k(\tau)[i]$ for every i, and hence $q(\tau)[i] = q_k(\tau)[i]$ holds. Hence if $K^{q_k}\colon q_k \to \omega$ witnesses that $q_k \in \mathcal{S}$ and we define $K\colon q \to \omega$ by $K(\nu) = K^{q_k}(\tau)$, where k is minimal such that ν has an extension in F_k and τ is the leftmost such one, then we see that K is a witness for $q \in \mathcal{S}$. ∎

The following corollary shows that below every nonzero degree of any set in $\mathbf{L}^{\mathcal{S}}$ lies a minimal nonzero degree of a real. Its proof follows immediately from that of Lemma 3.

COROLLARY 5: *Suppose that $p_0 \in \mathcal{S}, \alpha$ is an infinite ordinal and $\dot{\chi}$ is an $\mathcal{S}$-name for a new subset of α, i.e.*

$$p_0 \Vdash \dot{\chi} \in 2^\alpha \setminus V.$$

There exist $q \in \mathcal{S}, q \leqslant p_0$, a refining sequence $\langle F_k : k < \omega \rangle$ of fronts in q, a sequence $\langle X_k : k < \omega \rangle$ of disjoint finite subsets of α and a sequence $\langle \xi_\sigma : \sigma \in \bigcup_{k<\omega} F_k \rangle$ such that the following hold:

(1) *$q(\sigma) \Vdash \dot{\chi} \restriction \bigcup_{i<k} X_i = \xi_\sigma$, for every $\sigma \in F_k$;*
(2) *the family $\langle \xi_\tau : \tau \in F_{k+1} \cap q(\sigma) \rangle$ is pairwise incompatible, for every $\sigma \in F_k$.*

Letting $\dot{g}$ be the name for the $\mathcal{S}$-generic real, we conclude that

$$p_0 \Vdash \dot{g} \in V[\dot{\chi}],$$

and also

$$p_0 \Vdash \dot{\chi} \restriction X \notin V,$$

where $X = \bigcup_{k<\omega} X_k$ and thus $\dot{\chi} \restriction X$ is a name for a real.

Definition: Suppose that $\overline{p} \in \mathcal{S}^n$, $\dot{x}$ is an $\mathcal{S}^n$-name such that $\overline{p} \Vdash \dot{x} \in 2^\omega \setminus V$. We say that coordinate $i < n$ is **active** for $\overline{p}$ and $\dot{x}$, if the following holds:

(A) For every $\overline{p}' \leqslant \overline{p}$ there exist $\overline{q}^j \leqslant \overline{p}'$ and $\xi_j \in 2^{<\omega}$ for $j < 2$, so that $q^0_k = q^1_k$ for all $k \neq i, \xi_0$ and ξ_1 are incompatible and $\overline{q}^j \Vdash \xi_j \subseteq \dot{x}$.

Coordinate i not being active for $\overline{p}$ and $\dot{x}$ is certainly equivalent to saying:

(B) There exists $\overline{p}' \leqslant \overline{p}$ such that for all $\mathcal{S}^n$-generic $\overline{g}^j \in (2^\omega)^n, j < 2$, with $g^0_k = g^1_k$ for all $k \neq i$ and $g^j_k \in [p'_k]$ for all $k < n$ and $j < 2$ we have $\dot{x}[\overline{g}^0] = \dot{x}[\overline{g}^1]$.

If (B) holds we say that i is **passive** for $\overline{p}'$ and $\dot{x}$.

The following two lemmas hold as well for many other forcings with compact and perfect trees that satisfy Corollary 2.

LEMMA 6: *Suppose that $\overline{p} \in \mathcal{S}^n$ and $\dot{x}$ is an $\mathcal{S}^n$-name such that $\overline{p} \Vdash \dot{x} \in 2^\omega \setminus V$. Suppose further that $\langle n_k : k < \omega \rangle$ is increasing in ω and $\langle \xi_{\overline{\sigma}} : \overline{\sigma} \in \bigcup_{k<\omega} \overline{p} \restriction n_k \rangle$ is a family in $2^{<\omega}$ such that $\overline{p}(\overline{\sigma}) \Vdash \dot{x} \restriction k = \xi_{\overline{\sigma}}$ for every $\overline{\sigma} \in \overline{p} \restriction n_k$. If coordinate i is active for $\overline{p}$ and $\dot{x}$, then for every $\overline{p}' \leqslant \overline{p}$ and $m, r < \omega$ there exist $\overline{q} \leqslant \overline{p}'$ and $k < \omega$ such that $\overline{q} \in \mathcal{S}^n$ and*

(1) *$q_i \restriction r = p'_i \restriction r$;*

(2) $|q_i(\nu) \upharpoonright n_k| \geqslant m$ *for each* $\nu \in q_i \upharpoonright r$;

(3) *for every* $\overline{\sigma}, \overline{\tau} \in \overline{q} \upharpoonright n_k$ *such that* $\overline{\sigma} \upharpoonright n \setminus \{i\} = \overline{\tau} \upharpoonright n \setminus \{i\}$ *but* σ_i *and* τ_i *are incompatible we have that* $\xi_{\overline{\sigma}}$ *and* $\xi_{\overline{\tau}}$ *are incompatible.*

Proof: We may assume that $p'_k \upharpoonright r$ has just one element for every $k \neq i$. And, actually, each $q_k \upharpoonright n_k$ we are going to construct will still have just one element (for $k \neq i$). Let $p'_i \upharpoonright r = \{\zeta_j : j < s\}$. By using (A) repeatedly we build $\langle \overline{\sigma}(\nu) : \nu \in 2^{\leqslant m \cdot s} \rangle$ such that

(i) $\overline{\sigma}(\emptyset) \in \overline{p}' \upharpoonright r$ and $\overline{\sigma}(\emptyset)_i = \zeta_0$;

(ii) for every $l > 0$ there is k such that $\overline{\sigma}(\nu) \in \overline{p}' \upharpoonright n_k$ for every $\nu \in 2^l$;

(iii) $\xi_{\overline{\sigma}(\nu^\frown 0)}$ and $\xi_{\overline{\sigma}(\nu^\frown 1)}$ are incompatible for every ν;

(iv) if $\nu \subsetneqq \mu$ then $\overline{\sigma}(\mu)$ extends $\overline{\sigma}(\nu)$ coordinatewise, and if $|\nu| = |\mu|$ then $\overline{\sigma}(\nu) \upharpoonright n \setminus \{i\} = \overline{\sigma}(\mu) \upharpoonright n \setminus \{i\}$.

Let $\overline{\sigma}_1 = \overline{\sigma}(\nu) \upharpoonright n \setminus \{i\}$ for any $\nu \in 2^{m \cdot s}$. We repeat the construction above $\overline{\sigma}_1$ and ζ_1, producing a new $\langle \overline{\sigma}(\nu) : \nu \in 2^{\leqslant m \cdot s} \rangle$ and corresponding $\xi_{\overline{\sigma}(\nu)}$. Let $\overline{\sigma}_2 = \overline{\sigma}(\nu) \upharpoonright n \setminus \{i\}$ for any $\nu \in 2^{m \cdot s}$ and keep going above $\overline{\sigma}_2$ and ζ_2 etc. Thus we end with some $\overline{\sigma}_s$ and a finite n-dimensional tree $\langle \overline{\sigma}(\nu) : \nu \in 2^{\leqslant m \cdot s} \rangle$ with its terminal nodes of some length n_k. Thus $\overline{\sigma}_s = \overline{\sigma}(\nu) \upharpoonright n \setminus \{i\}$ for any $\nu \in 2^{m \cdot s}$. All the earlier finite trees from steps $j < s$ had their terminal nodes of some shorter length. In coordinates $n \setminus \{i\}$ these are all equal to $\overline{\sigma}_j$ (in step j), so here they are extended to level n_k by $\overline{\sigma}_s$, whereas in coordinate i no further extension took place. That is why now we extend the ith coordinate of each terminal node of the jth tree to level n_k by just one sequence inside p'_i (actually $p'_i(\zeta_j)$).

Let S_j be the set of all these sequences. We can now describe q_k for $k \in n \setminus \{i\}$. It is $\overline{p}_k((\overline{\sigma}_s)_k)$, where $(\overline{\sigma}_s)_k$ is coordinate k of $\overline{\sigma}_s$ of course. For $\zeta \in p'_i \upharpoonright n_k$ let $\overline{\sigma}_s^\frown \zeta$ be the element of $\overline{p}' \upharpoonright n_k$ with ith coordinate ζ and projection to $n \setminus \{i\}$ being $\overline{\sigma}_s$. The main point of the construction is that now for every $j < s$, if ζ, ζ' are different elements of S_j, then $\xi_{\overline{\sigma}_s^\frown \zeta}$ and $\xi_{\overline{\sigma}_s^\frown \zeta'}$ are incompatible of length k. As each S_j has size $2^{m \cdot s} > m \cdot s$, we can choose $S'_j \subseteq S_j$ for every $j < s$ such that $|S'_j| = m$ and $S'_j \cap S'_{j'} = \emptyset$ for different j, j' and hence $\xi_{\overline{\sigma}_s^\frown \zeta}$ and $\xi_{\overline{\sigma}_s^\frown \zeta'}$ are incompatible for any $\zeta \in S'_j$ and $\zeta' \in S'_{j'}$. Now we let $q_i = \bigcup \{\overline{p}'_i(\zeta) : \zeta \in \bigcup_{i<s} S'_j\}$ and thus got $\overline{q}$ as described. ∎

Lemma 7: *Suppose that* $\overline{p}, \dot{x}, \langle n_k : k < \omega \rangle$ *and* $\langle \xi_{\overline{\sigma}} : \overline{\sigma} \in \bigcup_{k<\omega} \overline{p} \upharpoonright n_k \rangle$ *are as in Lemma 6. For every* $\overline{p}' \leqslant \overline{p}$ *and* $r < \omega$ *there exist* $\overline{q} \leqslant \overline{p}'$ *and* $n_k \geqslant r$ *such that* $\overline{q} \in \mathcal{S}^n$, $\overline{q} \upharpoonright r = \overline{p}' \upharpoonright r$ *and for every* $\overline{\sigma}, \overline{\tau} \in \overline{q} \upharpoonright n_k$, *if* $\sigma_i \neq \tau_i$, *then* $\xi_{\overline{\sigma}} \neq \xi_{\overline{\tau}}$ *(hence* $\xi_{\overline{\sigma}}$ *and* $\xi_{\overline{\tau}}$ *are incompatible).*

Proof: Let $\langle \overline{\zeta}^j : j < s\rangle$ list $\prod_{l\in n\setminus\{i\}} p'_l \restriction r$ and let $\langle \zeta_j : j < t\rangle$ list $p'_i \restriction r$. By using Lemma 6 repeatedly it is straightforward to construct $\langle \overline{p}^j : j < s\rangle$ and $\langle k(j) : j < s\rangle$ such that

(1) $r < n_{k(0)}$ and $k(j) < k(j+1)$;

(2) $\overline{p}^{j+1} \leqslant \overline{p}^j \leqslant \overline{p}'$;

(3) $\overline{p}^j \restriction r = \overline{p}' \restriction r$,
$|p^j_l \restriction n_{k(j)}| = |p'_l \restriction r|$ for every $l \in n \setminus \{i\}$,
$|p^0_i(\zeta) \restriction n_{k(0)}| = t \cdot s^2$ for every $\zeta \in p'_i \restriction r$ and
$p^j_i \restriction n_{k(0)} = p^0_i \restriction n_{k(0)}$ and
$|p^j_i \restriction n_{k(j)}| = |p^0_i \restriction n_{k(0)}|$ for every $0 < j < s$;

(4) if $\overline{\eta}$ is the (by (3)) unique member of $\prod_{l\in n\setminus\{i\}} p^j_l(\zeta^j_l) \restriction n_{k(j)}$ and μ, ν are different elements of $p^j_i \restriction n_{k(j)}$, then $\xi_{\overline{\eta}^\frown \mu} \neq \xi_{\overline{\eta}^\frown \nu}$.

It is now clear that we can select $\vartheta_j \in p^{s-1}_i(\zeta_j) \restriction n_{k(s-1)}$ for each $j < s$ such that, letting $\overline{q}$ determined by $q_i = \bigcup\{p^{s-1}_i(\vartheta_j) : j < s-1\}$ and $q_l = p^{s-1}_l$ for $l \in n \setminus \{i\}, \overline{q}$ has the desired property. ∎

Remark: Note that Lemma 7 implies that under the same hypothesis and given $m < \omega$ there exists $\overline{q}$ and n_k such that the conclusion holds and in addition we have $|q_i(\sigma) \restriction n_k| > m$ for every $\sigma \in p'_i \restriction r$. Indeed, simply choose $r' \geqslant r$ so large that $p'_i(\sigma) \restriction r' > m$ for every $\sigma \in p'_i \restriction r$ and apply the lemma to r' instead of r. Certainly such r' exists, as every member of $\mathcal{S}$ is perfect.

MAIN LEMMA 8: *Suppose that $\overline{p} \in \mathcal{S}^n$ and $\overline{p} \Vdash_{\mathcal{S}^n} \dot{x} \in 2^\omega \setminus V$ for some $\mathcal{S}^n$-name $\dot{x}$, and that $\langle n_k : k < \omega\rangle$ is increasing in ω and $\langle \xi_{\overline{\sigma}} : \overline{\sigma} \in \bigcup_{k<\omega} \overline{p} \restriction n_k\rangle$ is such that $\overline{p}(\overline{\sigma}) \Vdash_{\mathcal{S}^n} \dot{x} \restriction k = \xi_{\overline{\sigma}}$ holds whenever $\overline{\sigma} \in \overline{p} \restriction n_k$. Suppose further that coordinate $i < n$ is active for $\overline{p}$ and $\dot{x}$ and that $r_0 < \omega$ is arbitrary. Then there exist $\overline{q} \leqslant \overline{p}$, $n_k > r_0$ and a front F of q_i such that the following properties are satisfied:*

(1) *$\overline{q} \restriction r_0 = \overline{p} \restriction r_0$;*

(2) *$\forall k < n\ \forall j < \omega\ q_k[j] = p_k[j]$;*

(3) *for every $\sigma \in F$ we have $r_0 \leq |\sigma| \leq n_k$ and for every $\overline{\sigma}, \overline{\tau} \in \overline{q} \restriction n_k$, if $F \restriction \sigma_i \neq F \restriction \tau_i$ then $\xi_{\overline{\sigma}} \neq \xi_{\overline{\tau}}$.*

Proof: The proof is a combination of the proofs of Lemmas 3 and 7. We shall work in dimension $n = 2$ and assume that coordinate 0 is active. This will help in keeping our presentation and notation reasonably clear. Then the generalization to arbitrary n is straightforward. We shall remark on this at the end of the proof. The proof will break down into three steps, each of them consisting of three further substeps. The three main steps are analogous to those in Lemma 3, the substeps are needed to guarantee property 2 in coordinate $k = 1$.

STEP 1: Find $\overline{q}^1 \leqslant \overline{p}, n_{k(1)} > r_0$ and $\tau \in q_0^1(st(p_0)^\frown 0) \restriction n_{k(1)}$ such that (1) and (2) hold and (*) for all $\sigma, \sigma' \in q_0^1(st(p_0)^\frown 1) \restriction n_{k(1)}$ with $\sigma \neq \sigma'$, for all $\mu, \nu \in q_1^1 \restriction n_{k(1)}$ we have $\xi_{(\sigma,\mu)} \neq \xi_{(\tau,\nu)}$ and $\xi_{(\sigma,\mu)} \neq \xi_{(\sigma',\nu)}$.

STEP 2: Find $\overline{q}^2 \leqslant \overline{q}^1$, $n_{k(2)} > n_{k(1)}$ such that
(1) $\overline{q}^2 \restriction n_{k(1)} = \overline{q}^1 \restriction n_{k(1)}$
(2) $\forall i \forall j q_i^2[j] = q_i^1[j]$
and for every $\varrho, \varrho' \in q_0^2(st(p_0^\frown 0)) \restriction n_{k(2)}$ such that $\varrho \neq \varrho'$ and ϱ and ϱ' is incompatible with τ, for every $\sigma, \sigma' \in q_0^2(st(p_0)^\frown 1) \restriction n_{k(2)}$ with $\sigma \neq \sigma'$ and for every $\mu, \nu \in q_1^2 \restriction n_{k(2)}$ we have $\xi_{(\varrho,\mu)} \neq \xi_{(\sigma,\nu)}$, $\xi_{(\varrho,\mu)} \neq \xi_{(\varrho',\nu)}$ and $\xi_{(\sigma,\mu)} \neq \xi_{(\sigma',\nu)}$.

STEP 3: Find $\overline{q}^3 \leqslant \overline{q}^2, n_{k(3)} > n_{k(2)}$ such that
(1) $\overline{q}^3 \restriction n_{k(2)} = \overline{q}^2 \restriction n_{k(2)}$,
(2) $\forall i \forall j q_i^3[j] = q_i^2[j]$
and for every $\tau', \tau'' \in q_0^3(\tau) \restriction n_{k(3)}$ with $\tau' \neq \tau''$, for every $\varrho \in q_0^3(st(p_0^\frown 0)) \restriction n_{k(3)}$ that is incompatible with τ, and for every $\mu, \nu \in q_1^3 \restriction n_{k(3)}$ we have $\xi_{(\tau',\mu)} \neq \xi_{(\varrho,\nu)}$ and $\xi_{(\tau',\mu)} \neq \xi_{(\tau'',\nu)}$.

Proof of Step 1: Apply Lemma 7 to $(p_0(st(p_0)^\frown 1), p_1(st(p_1)^\frown 1))$ with $r = \max\{K^{p_0}(st(p_0)^\frown 0), K^{p_1}(st(p_1)^\frown 0), r_0\}$, getting $\overline{q}', n_k^0$ such that $|q_0'(\sigma) \restriction n_k^0| > |p_1(st(p_1)^\frown 1) \restriction r|$ for every $\sigma \in p_0(st(p_0)^\frown 1) \restriction r$ (see the remark after Lemma 7). Fix $\tau_0 \in p_0(st(p_0)^\frown 0) \restriction n_k^0$. Now we can easily prune q_0' so that for each $\sigma \in p_0(st(p_0)^\frown 1) \restriction r$ we keep just one $\sigma' \in q_0'(\sigma) \restriction n_k^0$ in a way that for all $\nu, \mu \in q_1' \restriction n_k$ we have $\xi_{\langle\tau_0,\nu\rangle} \neq \xi_{\langle\sigma',\mu\rangle}$. Thus we got $\overline{q}'' \leqslant \overline{q}'$ such that $q_1'' = q_1'$ and $q_0'' = \bigcup\{q_0'(\sigma') : \sigma \in p_0(st(p_0)^\frown 1) \restriction r\}$. We certainly may assume that $|q_1' \restriction n_k| \geqslant 2$. Fix $\nu \in q_1' \restriction n_k$. Define a new r to be $K_{q_1'}(\nu)$. Let $\overline{p}'$ be defined by $p_0' = p_0(st(p_0)^\frown 0) \cup q_0''$ and $p_1' = p_1(st(p_1)^\frown 0) \cup q_1'$.

Now apply Lemma 7 again to $(p_0'(st(p_0^\frown 1)), p_1(st(p_1)^\frown 0) \cup \bigcup\{q_1'(\mu) \colon \mu \in q_1' \restriction n_k^0 \setminus \{\nu\}\})$ with the new r. Thus we get n_k^1 and a new $\overline{q}'$ such that $|q_0'(\sigma') \restriction n_k^1| > |p_1' \restriction r|$ holds for every $\sigma' \in p_0'(st(p_0)^\frown 1)$ of length n_k^0. Fix $\tau_1 \in p_0(\tau_0) \restriction n_k^1$ and then prune q_0' so that for each $\sigma \in p_0'(st(p_0)^\frown 1) \restriction r$ we keep just one $\sigma' \in q_0'(\sigma) \restriction n_k^1$ in a way that for all $\mu, \zeta \in q_1' \restriction n_k^1$ we have $\xi_{(\tau_1,\mu)} \neq \xi_{(\sigma',\zeta)}$. Thus we got $\overline{q}'' \leqslant \overline{q}'$ such that $q_1'' = q_1'$ and $q_0'' = \bigcup\{q_0'(\sigma') \colon \sigma \in p_0'(st(p_0)^\frown 1) \restriction r\}$. We define $\overline{p}''$ by letting $p_0'' = p_0(st(p_0)^\frown 0) \cup q_0''$ and $p_1'' = q_1' \cup p_1'(\nu)$.

In order to complete Step 1 we must do a third round of decoupling, this time of the trees $p_1''(st(p_1)^\frown 0)$ and $p_1''(\nu)$. For this fix $\mu \in q_1' \restriction n_k^1$ and define a new r to be $K^{p''}(\mu)$. Apply Lemma 7 to $(p_0''(st(p_0)^\frown 1), p_1''(st(p_1)^\frown 0) \cup p_1''(\nu))$

and the new r. We get new n_k^2 and a new $\overline{q}'$ such that $|q_0'(\sigma') \restriction n_k^2| > |p_1'' \restriction r|$ holds for every $\sigma' \in p_0''(st(p_0)^\frown 1)$ of length n_k^1. Fix $\tau_2 \in p_0(\tau_1)$ of length n_k^2 and then prune q_0' as before so that for each $\sigma \in p_0''(st(p_0)^\frown 1) \restriction r$ we keep just one $\sigma' \in q_0'(\sigma) \restriction n_k^2$ in a way that for all $\zeta, \varrho \in q_1' \restriction n_k^2$ we have $\xi_{(\tau_2,\zeta)} \neq \xi_{(\sigma',\varrho)}$. Thus we got $\overline{q}'' \leqslant \overline{q}'$ with $q_1'' = q_1'$ and $q_0'' = \bigcup\{q_0'(\sigma')\colon \sigma \in p_0''(st(p_0)^\frown 1) \restriction r\}$. Finally we define $\overline{p}'''$ by letting $p_0''' = p_0(st(p_0)^\frown 0) \cup q_0''$ and $p_1''' = q_1' \cup \bigcup\{p_1''(\varrho)\colon \varrho \in p_1''(st(p_1)^\frown 1) \restriction |\nu| \setminus \{\nu\}\}$. Then we can let $\tau = \tau_2$ and $\overline{q}^1 = \overline{p}'''$ and $n_{k(1)} = n_k^2$, and these will suffice for Step 1. $\blacksquare_{Step 1}$

Proof of Step 2: This proof is somewhat simpler than that of Step 1, again it uses the idea of dividing q_1^1 into three areas and decoupling any two of them in three substeps, each time using the third one to guarantee property (2). Pick $\nu \in q_1^1(st(p_1)^\frown 1) \restriction n_{k(1)}$ arbitrarily. Then the cone above $st(p_1)^\frown 0$ is the first area, the cone above ν the second one, and everything above $st(p_1)^\frown 1$ incompatible with ν is the third one. Let $r = \max\{K^{q_0^1}(\tau_2), K^{q_1^1}(st(p_1)^\frown 0)\}$. Apply Lemma 7 to $(q_0^1(st(p_0)^\frown 1) \cup \bigcup\{q_0^1(\varrho)\colon \varrho \in q_0^1(st(p_0)^\frown 0) \restriction n_{k(1)} \setminus \{\tau\}\}, q_1^1(st(p_1)^\frown 1))$ with r as above and obtain $\overline{q}'$ and $n_k^0 > r$. Define $\overline{p}'$ by letting $p_0' = q_0^1(\tau) \cup q_0'$ and $p_1' = q_1^1(st(p_1)^\frown 0) \cup q_1'$. Now let $r = \max\{n_k^0, K^{p_1'}(\nu)\}$, apply Lemma 7 to $(q_0', \bigcup\{p_1'(\mu)\colon \mu \in p_1' \restriction n_{k(1)} \setminus \{\nu\}\})$ with this r and obtain $n_k^1 > r$ and a new $\overline{q}'$. Define $\overline{p}''$ by $p_0'' = q_0^1(\tau) \cup q_0'$ and $p_1'' = q_1' \cup p_1'(\nu)$. Finally pick any $\mu \in q_1^1 \restriction n_{k(1)} \setminus \{\nu\}$ and let $r = \max\{n_k^1, K^{p_1''}(\mu)\}$. Apply Lemma 7 to $(q_0', p_1''(st(p_0)^\frown 0) \cup p_1''(\nu))$ with this last r, and obtain n_k^2 and a new $\overline{q}'$. Now we can define $\overline{q}^2$ by $q_0^2 = q_0^1(\tau) \cup q_0'$ and $q_1^2 = q_1' \cup \bigcup\{p_1''(\zeta)\colon \zeta \in q_1^1 \restriction n_{k(1)} \setminus \{\nu\}\}$, and we define $n_{k(2)} = n_k^2$. Then these objects are as described for Step 2. $\blacksquare_{Step 2}$

Proof of Step 3: This is now completely analogous to Step 2. First we choose $\sigma \in q_0^2(st(p_0)^\frown 1) \restriction n_{k(2)}$ and let $r = K^{q_0^2}(\sigma)$. This time we leave $q_0^2(st(p_0)^\frown 1)$ unchanged so that (2) will hold, and we work inside $(q_0^2(st(p_0)^\frown 0), q_1^2)$ above level r in three substeps to get the desired $\overline{q}^3$ and $n_{k(3)}$. $\blacksquare_{Step 3}$

Now we let $\overline{q} = \overline{q}^3$, $n_k = n_{k(3)}$ and

$$F = q_0^3(stem(p_0)^\frown 0) \restriction n_{k(3)} \cup q_0^3(stem(p_0)^\frown 1) \restriction n_{k(2)}.$$

Then these objects as described in the Main Lemma. It should be clear that generalizing this proof to arbitrary finite dimension n is straightforward. Indeed, we perform the same three main steps, but each of them is now broken up into 3^{n-1} substeps. This is because now in each coordinate different from i we divide into three areas and in one substep we handle one independent choice of two of them in each coordinate. $\blacksquare$

Corollary 9: *Suppose that $\overline{p}, \dot{x}, n_k, \xi_{\overline{\sigma}}, r$ and i are as in the Main Lemma. There exist $\overline{q} \leqslant \overline{p}$, an increasing sequence $\langle k(l) : l < \omega\rangle$ in ω and a sequence $\langle F_l : l < \omega\rangle$ of fronts in q_i such that the following hold:*

(1) $\overline{q} \restriction r = \overline{p} \restriction r$;
(2) $\forall k < n \forall j < \omega q_k[j] = p_k[j]$;
(3) *every* $\sigma \in F_l$ *has length at least* $n_{k(l)}$ *and at most* $n_{k(l+1)}$ *(hence* F_{l+1} *refines* F_l*);*
(4) *for every* $l < \omega$ *and for every* $\overline{\sigma}, \overline{\tau} \in \overline{q} \restriction n_{k(l+1)}$, *if* $F_l \restriction \sigma_i \neq F_l \restriction \tau_i$ *then* $\xi_{\overline{\sigma}} \neq \xi_{\overline{\tau}}$.

Theorem 10: (a) *The lattice of constructibility degrees of reals in a forcing extension* $\mathbf{L}[\overline{g}]$ *of* $\mathbf{L}$ *by* $\mathcal{S}^n$ *(so* $\overline{g} \in (2^\omega)^n$ *is* $\mathcal{S}^n$*-generic over* $\mathbf{L}$*) is the set of all* $\langle g_i : i \in E\rangle$ *where* $E \subseteq n$*; hence it is isomorphic to* $\mathcal{P}(n)$.

(b) *Given any Borel function* $f\colon \prod_{i<n}[p_i] \to \mathbb{R}$, *where* $\overline{p} \in \mathcal{S}^n$, *there exist* $\overline{q} \leqslant \overline{p}$ *and* $E \subseteq n$ *such that for all* $\overline{x}, \overline{y} \in \prod_{i<n}[q_i] f(\overline{x}) = f(\overline{y})$ *iff* $\pi_E(\overline{x}) = \pi_E(\overline{y})$. *Here* π_E *is the generalized projection defined by* $\pi_E(\overline{x}) = \{x_i : i \in E\}$.

Proof: (a) We work in $\mathbf{L}$. Let $\dot{x} \in \mathbf{L}$ be an $\mathcal{S}^n$-name such that $\Vdash \dot{x} \in 2^\omega \setminus \mathbf{L}$. It is straightforward to construct $\overline{p} \in \mathcal{S}^n$ such that every $i < n$ is either active or passive for $\overline{p}$ and $\dot{x}$. Let A be the set of all active $i < n$. We now apply Corollary 9 repeatedly to each $i \in A$ and end up with $\overline{q} \leqslant \overline{p}$, increasing sequences $\langle k(l) : l < \omega\rangle, \langle n_{k(l)} : l < \omega\rangle$ in ω and sequences $F = \langle F^i_l : i \in A, l < \omega\rangle$ and $X = \langle \xi_{\overline{\sigma}}\colon \overline{\sigma} \in \bigcup_{l<\omega} q \restriction n_{k(l)}\rangle$ such that F^i_l is a front of q_i with every $\sigma \in F^i_l$ of length at least $n_{k(l)}$ and at most $n_{k(l+1)}$ and for every $\overline{\sigma}, \overline{\tau} \in q \restriction n_{k(l+1)}$ we have $\overline{q}(\overline{\sigma}) \Vdash \xi_{\overline{\sigma}} = \dot{x} \restriction k(l)$ and if $i \in A$ and $F^i_l \restriction \sigma_i \neq F^i_l \restriction \tau_i$ then $\xi_{\overline{\sigma}} \neq \xi_{\overline{\tau}}$. On the other hand, if $\overline{\sigma} \restriction A = \overline{\tau} \restriction A$ then $\xi_{\overline{\sigma}} = \xi_{\overline{\tau}}$ by passivity of the coordinates outside A. If we now let $\overline{g} \in \prod_{i<n}[q_i]$ be $\mathcal{S}^n$-generic over $\mathbf{L}$, then $\dot{x}[\overline{g}] = \bigcup_{l<\omega} \xi_{\overline{g}\restriction n_{k(l)}}$. But by the last remark, $\dot{x}[\overline{g}]$ is determined by $\overline{g} \restriction A$. Conversely, by knowing $\bigcup_{l<\omega} \xi_{\overline{g}\restriction n_{k(l)}}$, F and X we can reconstruct $\overline{g} \restriction A$. Hence $\dot{x}[\overline{g}]$ determines the same degree as $\overline{g} \restriction A$.

(b) Let $(N, \in)$ be a countable transitive model of ZF^- such that $f, \overline{p} \in N$. We work in N first. Let $\dot{\overline{g}}$ be the canonical $\mathcal{S}^n$-name for the $\mathcal{S}^n$-generic real. Choose $\overline{p}' \leqslant \overline{p}$ such that either $\overline{p}'$ decides the value of $f(\dot{\overline{g}})$ in N or $\overline{p}' \Vdash f(\dot{\overline{g}}) \in 2^\omega \setminus N$. In the first case let $\overline{q} = \overline{p}'$, in the second case find $\overline{p}'' \leqslant \overline{p}'$ and $A \subseteq n$ such that A is the set of active coordinates for $\overline{p}''$ and $f(\dot{\overline{g}})$ and then apply Corollary 9 to get $\overline{q} \leqslant \overline{p}''$ together with the $n_{k(l)}$, $\xi_{\overline{\sigma}}$ and F^i_l as in (a). Now apply the remark after Corollary 2 and find $\overline{q}' \leqslant \overline{q}, \overline{q}'$ now in $V \setminus N$, such that every $\overline{x} \in \prod_{i<n}[q'_i]$ is $\mathcal{S}^n$-generic over N. By absoluteness we now have that in the first

case $f \restriction \prod_{i<n}[q_i']$ is constant and in the second case $f\overline{x} = f\overline{y}$ iff $\overline{x} \restriction A = \overline{y} \restriction A$ for all $\overline{x}, \overline{y} \in \prod_{i<n}[q_i']$. ∎

3. Splitting cubes

In this concluding section we give a negative answer to Zapletal's question and we show that there are limitations to a reasonable canonization modulo splitting cubes.

Definition: Let $n < \omega$ and let f be any function with domain $\prod_{i<n}[p_i]$ for some $\overline{p} \in \mathcal{S}^n$. We say that $\overline{p}$ is **canonical** for f if there exists $E \subseteq n$ such that $\forall \overline{x}, \overline{y} \in \prod_{i<n}[p_i] f\overline{x} = f\overline{y} \Leftrightarrow \pi_E\overline{x} = \pi_E\overline{y}$.

THEOREM 11: *Suppose that $f\colon (2^\omega)^n \to 2^\omega$ is Borel, $p \in \mathcal{S}$ and $r < \omega$. There exist $q \leqslant p$ and an increasing sequence $\langle m_k : k < \omega\rangle$ in ω such that $q \restriction r = p \restriction r, m_0 = r$ and for every one-to-one sequence $\overline{\sigma} \in \prod_n q \restriction m_k$, for every $\overline{\tau} \in \prod_{i<n} q(\sigma_i) \restriction m_{k+1}$ we have that $\langle q(\tau_i) : i < n\rangle$ is canonical for f.*

Proof: By Theorem 10 (b) the set $D = \{\overline{q} \in \mathcal{S}^n : \overline{q}$ is canonical for $f\}$ is dense in $\mathcal{S}^n$. By applying Lemma $1'$ repeatedly it is straightforward to construct a descending chain $\langle q_k : k < \omega\rangle$ in $\mathcal{S}$ and an increasing sequence $\langle m_k : k < \omega\rangle$ in ω such that

(1) $m_0 = r, q_0 = p$;
(2) $q_{k+1} \restriction m_k = q_k \restriction m_k$ and $q_{k+1}(\sigma)[j] = q_k(\sigma)[j]$ for every $\sigma \in q_k \restriction m_k$ and $j \geqslant m_k$;
(3) for every $\overline{\sigma} \in \prod_n q_k \restriction m_k$, for every $\overline{\tau} \in \prod_{i<n} q_k(\sigma_i) \restriction m_{k+1}$ we have that $\langle q_{k+1}(\tau_i) : i < n\rangle \in D$.

If we let q be the downwards closure of $\bigcup\{q_k \restriction m_k : k < \omega\}$, equivalently $q = \bigcap_{k<\omega} q_k$, q and $\langle m_k : k < \omega\rangle$ are as desired. ∎

LEMMA 12: *Suppose that $f\colon [2^\omega]^n \to 2^\omega$ is Borel, $p \in \mathcal{S}, r \in \omega$ and $C \subseteq 2^\omega$ is perfect. There exist $q \leqslant p$ and a perfect $C' \subseteq C$ such that $q \restriction r = p$, for every $\sigma \in p \restriction r$ and for every $j \geqslant r q(\sigma)[j] = p(\sigma)[j]$, and for every increasing $\overline{\sigma} \in \prod_n p \restriction r$, if $\langle p(\sigma_i) : i < n\rangle$ is canonical for f then $f\overline{x} \notin C'$ for every $\overline{x} \in \prod_{i<n}[q(\sigma_i)]$.*

Proof: For each $\sigma \in p \restriction r$ we fix a splitmode $\nu_\sigma \in p(\sigma)$ of minimal length. Let $K^* = \max\{K^p(\nu_\sigma^\frown 0)\colon \sigma \in p \restriction r\}$. We declare $q \restriction K^* = p \restriction K^*$ and $q(\nu_\sigma^\frown 0) = p(\nu_\sigma^\frown 0)$ for every $\sigma \in p \restriction r$. Thus $q(\sigma)[j] = p(\sigma)[j]$ will hold for every

$j \geqslant r$. Fix the first $\overline{\sigma}$ as in the lemma. If f is constant on $\prod_{k<n}[p(\sigma_k)]$ we choose perfect $C^1 \subseteq C$ not containing this constant value and go on to consider the next $\overline{\sigma}$. Otherwise we can fix (at least one) $i < n$ such that $f \restriction \prod_{k<n}[p(\sigma_k)]$ is one-to-one in coordinate i. Fix $\mu \in p(\nu_{\sigma_i}^\frown 1) \restriction K^*$. For $X \subseteq [p_i]$ let $B(X) = \prod_{k<n} X_k$ where

$$X_k = \begin{cases} X, & \text{if } k = i \\ [p(\sigma_k)], & \text{if } k \neq i. \end{cases}$$

If there exists $\mu' \in p(\mu)$ (extending μ) such that $C \cap ran f \restriction B([p(\mu')])$ is countable we choose perfect $C^1 \subset C$ disjoint from this countable set shrink $p(\mu)$ to $p(\mu')$, and go on to consider the next $\mu \in p(\nu_{\sigma_i}^\frown 1) \restriction K^*$. Otherwise we can find incompatible $\mu^j \in p(\mu), j < 2$ such that $C \cap ran f \restriction B([p(\mu^j)])$ is uncountable for both $j < 2$. As f is one-to-one in coordinate i these two sets are disjoint. As every uncountable analytic set contains a perfect subset (see [K]), we can choose perfect $C^1 \subseteq C \cap ran f \restriction B([p(\mu^0)])$. Now we shrink $p(\mu)$ to $p(\mu^1)$ and go on to consider the next $\mu \in p(\nu_{\sigma_i}^\frown 1) \restriction K^*$. In this way we take care of all $\mu \in p(\nu_{\sigma_i}^\frown 1) \restriction K^*$, and then proceed to take care of all $\overline{\sigma}$ as in the lemma, each time choosing $i < n$ such that $f \restriction \prod_{k<n}[\sigma_k]$ is one-to-one in coordinate i. During this (finite) process we are constructing a descending sequence of perfect sets and another one of splitting trees below p. If C' and p' are the last objects from these sequences we let $q(\nu_\sigma^\frown 1) = p'(\nu_\sigma^\frown 1)$ for every $\sigma \in p \restriction r$. Together with our earlier stipulations about q, this defines q. Then q and C' are our desired objects. ∎

Remark: It is easy to check that in Lemma 12 we can start from any finitely many perfect sets $C_0, \ldots, C_m$ and find perfect $C'_0 \subseteq C_0, \ldots, C'_m \subseteq C_m$ such that $f\overline{x} \notin C'_j$ for every $j \leqslant m$, for every $\overline{x}$ as there.

THEOREM 13: *Suppose that $f\colon [2^\omega]^n \to 2^\omega$ is Borel and $p \in \mathcal{S}$. There exists $q \leqslant p$ such that $f \restriction [q]^n$ does not attain all values in 2^ω, actually it misses a perfect set of them. Hence we have*

$$2^\omega \to_{Borel} [analytic\ splitting]^n_{2^{\aleph_0}}$$

for every $n < \omega$.

Proof: First we apply Theorem 11 and find $q^1 \leqslant p$ and an increasing sequence $\langle m_k : k < \omega \rangle$ as in the theorem. Then we apply Lemma 12 repeatedly to construct a descending chain $\langle q_k : k < m \rangle$ below q^1 and another one $\langle C_k : k < \omega \rangle$ consisting of perfect sets such that the following hold:

(1) $q_0 = q^1, q_{k+1} \restriction m_k = q_k \restriction m_k$;
(2) for every $\sigma \in q_k \restriction m_k$, for every $j \geqslant m_k$ we have $q_{k+1}(\sigma)[j] = q_k(\sigma)[j]$;
(3) for every increasing $\overline{\sigma} \in \prod_n q_k \restriction m_k$, if $\langle q_k(\sigma_i) : i < n\rangle$ is canonical for f, then $f\overline{x} \notin C_{k+1}$ for every $\overline{x} \in \prod_{i<n}[q_{k+1}(\sigma_i)]$.

Now we let $q \in \mathcal{S}$ be determined by $q \restriction m_k = q_k \restriction m_k$ for every k. Note that by compactness $\bigcap_{k<\omega} C_k \neq \emptyset$, so let α be an element of this set. Let $\overline{x} \in [q]^n$. Choose m_k large enough so that $\langle x_i \restriction m_k : i < n\rangle \in \prod_{n} {}_{<}q_k \restriction m_k$. By construction we know that $\langle q_{k+1}(x_i \restriction m_{k+1}) : i < n\rangle$ is canonical for f and hence $f\overline{x} \notin C_{k+2}$ and thus $f\overline{x} \neq \alpha$. It should be clear how one modifies this construction, using the remark after Lemma 12, to ensure that $\bigcap_{k<\omega} C_k$ is perfect. ∎

Theorem 11 can be viewed as a weak canonization modulo splitting cubes. Our concluding result shows that there exists a limitation for finding a neater such result in the style of Lefmann's canonization modulo perfect cubes (see [L]). His result can be obtained from the canonization modulo perfect blocks (the analog of our Theorem 10 (b)) together with not much more than the simple fact that every perfect tree whose splitnodes are coloured by any number of colours contains a perfect subtree with all its splitnodes either the same colour or of pairwise different colours. Our concluding result shows that this last fact fails for splitting trees.

THEOREM 14: *There exits $p \in \mathcal{S}$ and a coloring of the splitnodes of p by two colors such that for no $q \leqslant p, q \in \mathcal{S}$, all splitnodes of q have the same color.*

Proof: Inductively we construct an increasing sequence $\langle n_k : k < \omega\rangle$ and a family $\langle \nu_\sigma : \sigma \in 2^{<\omega}\rangle$ in $2^{<\omega}$ such that the following hold:

(1) If $|\sigma| = k$, then $|\nu_\sigma| = n_k$, and $\nu_\sigma \subseteq \nu_{\sigma^\frown i}$ for every $k < \omega, i < 2$;
(2) $(\nu_{\sigma^\frown 0}, \nu_{\sigma^\frown 1})$ is a splitting pair above ν_σ (see [Sp 2]), i.e. $\nu_{\sigma^\frown 0}(j) = 1 - \nu_{\sigma^\frown 1}(j)$ for every $j \in |\nu_{\sigma^\frown 0}| \setminus |\nu_\sigma|$;
(3) for every π: $2^k \to 2$ there exists $j \in n_{k+1} \setminus n_k$ such that for any $\sigma, \tau \in 2^k$ we have

$$\nu_{\sigma^\frown \pi(\sigma)}(j) = \nu_{\tau^\frown \pi(\tau)}(j).$$

The construction is straightforward. Now let p be the tree generated by the family $\langle \nu_\sigma : \sigma \in 2^{<\omega}\rangle$. Certainly we have $p \in \mathcal{S}$ by (2). Actually we can let $K^p(\nu_\sigma) = |\nu_\sigma|$ and $K^p(\nu) = K^p(\nu_\sigma)$ where ν_σ is the shortest sequence extending ν, for all other ν.

Moreover, the splitnodes of p are precisely all the ν_σ. Now color ν_σ by the parity of $|\sigma|$. Suppose that q is a subtree of p with all its splitnodes of the same color, without loss say of even color. This means that for every k there exists $\pi_k \colon 2^{2k+1} \to 2$ such that $q \restriction n_{2k+2}$ is contained in $\{\nu_{\sigma^\frown \pi(\sigma)} : \sigma \in 2^{2k+1}\}$. By (3) there exists $j \in n_{2k+2} \setminus n_{2k+1}$ such that $q(j)$ has only one element. As there are infinitely many such j, q is not a splitting tree. ∎

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